

Formation and Pinch-off of Viscous Droplets in the Absence of Surface Tension: an Exact Result

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Within a class of exact time-dependent non-singular N -logarithmic solutions [Mineev-Weinstein and Dawson, Phys. Rev. **E 50**, R24 (1994); Dawson and Mineev-Weinstein, Phys. Rev. **E 57**, 3063 (1998)], we have found solutions which describe the development and pinching off of viscous droplets in the Hele-Shaw cell in the absence of surface tension.

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The process of viscous fingering in the Hele-Shaw cell, in which a non-viscous fluid pushes a viscous one, has attracted the attention of many physicists and mathematicians since 1958 [1]. The pressure field in the viscous fluid is harmonic because of D'Arcy's law, $\mathbf{v} = -\nabla p$, incompressibility, and continuity, $\text{div } \mathbf{v} = 0$, where \mathbf{v} is fluid velocity. Because pressure is constant along the moving interface between the non-viscous fluid and the viscous fluid in the absence of surface tension, the time-dependent conformal technique was found to be of great help. The integro-differential equation

$$\text{Im}(\bar{z}_t z_\phi) = 1 \quad (1)$$

for the moving complex interface $z(t, \phi)$ was derived [2]. Here the bar denotes complex conjugation, z_t and z_ϕ are partial derivatives, and the map $z(t, \phi)$ is conformal for $\text{Im } \phi \leq 0$. This equation which we call the Laplacian growth equation (LGE) has many remarkable properties [3]- [5]. In this paper, we make use of a broad class of exact time-dependent non-singular N -logarithmic solutions in the channel geometry (where the interface moves between the two parallel walls of the Hele-Shaw cell [4]) and in the radial geometry (where the moving contour is closed [5]). The equation (1) is valid for both of these geometries provided the flux of fluid is constant. In the channel geometry, the N -logarithmic solution has the form [4]

$$z(t, \phi) = \tau(t) + i\phi + \sum_{k=1}^N \alpha_k \log(1 - a_k(t)e^{-i\phi}), \quad (2)$$

where $\alpha_k = \text{const}$, and $|a_k| < 1$. The time dependence of $a_k(t)$ and $\tau(t)$ is given by

$$\begin{cases} \beta_k = z(t, i \log \bar{a}_k) \\ = \tau - \log \bar{a}_k + \sum_{l=1}^N \alpha_l \log(1 - \bar{a}_k a_l) = \text{const}, \\ t + C = \left(1 - \frac{1}{2} \sum_{l=1}^N \alpha_l\right) \tau + \frac{1}{2} \sum_{l=1}^N \alpha_l \log(a_l), \end{cases} \quad (3)$$

where $k = 1, 2, \dots, N$ and C are constants in time. The Hele-Shaw cell width is chosen to be 2π . Hence $z(2\pi) - z(0) = 2\pi i$. This is for periodic boundary condition, while for the no-flux boundary conditions the same formulas (2-3) are valid, but the sums in (2-3) contain both terms with α_k and a_k and with $\bar{\alpha}_k$ and \bar{a}_k unless both α_k and a_k are real.

The N -logarithmic solution in radial geometry is

$$z(t, \phi) = r(t)e^{i\phi} + \sum_{k=1}^N \alpha_k \log\left(\frac{e^{i\phi}}{a_k(t)} - 1\right), \quad (4)$$

where $\sum_{k=1}^N \alpha_k = 0$, $\alpha_k = \text{const}$, and $|a_k| < 1$.

Time dependence of $a_k(t)$ and $r(t)$ is given by

$$\begin{cases} \beta_k = z(t, i \log \bar{a}_k) \\ = \frac{r}{\bar{a}_k} + \sum_{l=1}^N \alpha_l \log\left(\frac{1}{\bar{a}_k a_l} - 1\right) \\ = \text{const} \\ 2t + C = r^2 - r \sum_{k=1}^N \frac{\alpha_k}{a_k}. \end{cases}$$

For a broad range of $\{\alpha_k\}$'s (which we do not specify here) the solutions (2) are free of finite-time singularities (cusps) [4]- [5].

In this paper we present a subclass of solutions (2) which describes the change of topology of the moving interface which occurs in the process of droplet formation of the viscous fluid. The moving interface continues to evolve in accordance with the equation (1). We have found the analytic continuation from the domain immediately before the topological breakdown occurred to the domain immediately after the droplet formation. This continuation allows us to study the interface dynamics which includes an arbitrary number of topological changes of this kind in the absence of surface tension. We believe that inclusion of surface tension in these studies will shed light on which topological changes are physically allowed and which are forbidden. The goal of this article is to report a new class of solutions which describes droplet formation in the absence of surface tension.

Let us consider the following 3-logarithmic solution of the LGE (1) in channel geometry:

$$z(t, \phi) = \tau(t) + i\phi + \alpha \log \frac{e^{i\phi} - a(t)}{e^{i\phi} - b(t)}, \quad (5)$$

where $\alpha > 0$, $0 < b(0) < a(0) < 1$, and the third logarithmic singularity, corresponding to the term $i\phi$, is zero. It follows from (3) that the time evolution of $a(t)$ and $b(t)$ obeys the equations

$$\begin{cases} \beta_1 = t - \frac{\alpha}{2} \log \frac{a}{b} + \alpha \log \frac{1-a^2}{1-ab} - \log a \\ \beta_2 = t - \frac{\alpha}{2} \log \frac{a}{b} + \alpha \log \frac{1-ab}{1-b^2} - \log b \end{cases} \quad (6)$$

Both $a(t)$ and $b(t)$ reach the unit circle simultaneously at the moment, t^* , which is easy to calculate:

Since a and b are close to 1, then deviations of a and b from 1, defined as $p = 1 - a$ and $q = 1 - b$, are very small, that is $0 < p = 1 - a < q = 1 - b \ll 1$. Thus the last system in the leading order is

$$\begin{cases} \beta_1 = t^* + \alpha \log c - \alpha \log \frac{c+c^{-1}}{2} \\ \beta_2 = t^* + \alpha \log c + \alpha \log \frac{c+c^{-1}}{2} \end{cases} \quad (7)$$

where $c^2 = \lim_{t \rightarrow t^*} p/q$ is a finite constant (assuming that t^* is finite). The solution of (7) is

$$c = A - \sqrt{A^2 - 1} \quad (8)$$

$$t^* = \beta_1 - \alpha \log(1 - \sqrt{1 - e^{(\beta_1 - \beta_2)/\alpha}}), \quad (9)$$

where $A = e^{(\beta_2 - \beta_1)/2\alpha}$.

It is also easy to calculate the behavior of $a(t)$ and $b(t)$ immediately prior to the moment t^* :

Defining $\delta t = t^* - t$ and expanding $a(t)$ and $b(t)$ near t^*

$$\begin{cases} a = 1 - p_0 \delta t - p_1 (\delta t)^2 \\ b = 1 - q_0 \delta t - q_1 (\delta t)^2, \end{cases} \quad (10)$$

(clearly then, that $p_0/q_0 = c^2$), we rewrite (6) keeping linear terms with respect to δt :

$$\begin{cases} 1 = \alpha \left(\frac{p_1}{p_0} - \frac{p_1 + p_0}{q_1 + q_0} \right) + p_0 \left(\frac{\alpha}{2} \frac{q_0 - p_0}{q_0 + p_0} + 1 \right) \\ 1 = \alpha \left(\frac{p_1 + p_0}{q_1 + q_0} - \frac{q_1}{q_0} \right) + q_0 \left(\frac{\alpha}{2} \frac{q_0 - p_0}{q_0 + p_0} + 1 \right) \end{cases} \quad (11)$$

Keeping in mind that $c^2 = p_0/q_0$ (see above) we find from this last system that

$$\begin{cases} p_0 = \frac{c^2}{(1+\alpha/2)+c^2(1-\alpha/2)} \\ q_0 = \frac{1}{(1+\alpha/2)+c^2(1-\alpha/2)}, \end{cases} \quad (12)$$

where c is given by (9).

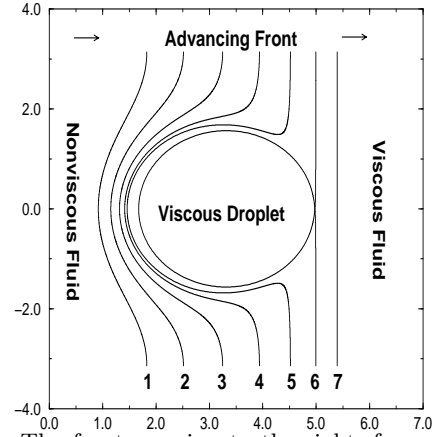


FIG. 1. The front, moving to the right, forms a viscous droplet, and then leaves it behind. The numbers 1-7 denote 7 distinct snapshots of the advancing front.

It is interesting to see how the advancing front moves in this case. Figure 1 shows that the initially almost planar interface ($b(0) \ll 1$) advances until a groove of width $\pi\alpha$ is created at the stagnation point with complex coordinate $\beta_1 - \alpha \log 2$. Then later this groove becomes thinner and thinner until at the moment t^* it disappears completely. Since both $a(t)$ and $b(t)$ reach the unit circle simultaneously, (5) contains a singular term $\log[(e^{i\phi} - 1)/(e^{i\phi} - 1)]$ which has a removable singularity at 1. This term equals zero *exactly at this moment of time*. If we formally consider $t > t^*$ in (6) we see that both $a(t)$ and $b(t)$ continue their evolution outside the unit circle. This is clearly an unphysical part of the solution (5), which violates the framework of validity of this solution, which demands that the exterior of the unit circle should be free of singularities except at infinity. We also note that the range of the variable ϕ parametrizing the moving curve, corresponding to the groove of viscous liquid which is left behind the advancing front, becomes negligibly small when the “neck” of this groove becomes very thin. This allows us to make the analytical continuation from the moment t_-^* immediately prior to the disappearance of the neck and pinching off the viscous droplet behind the moving interface to the moment t_+^* that is immediately after this event occurs. Once both singularities $a(t)$ and $b(t)$ reach the unit circle at $t = t^*$, the last term in (5) vanishes and the subsequent evolution ignores this term. The interface in this case becomes a planar front described by the equation

$$z(t, \phi) = \tau(t) + i\phi, \quad (13)$$

where $\tau = t$ in this case. In the physical plane, this analytical continuation at the moment $t = t^*$ shows that the viscous droplet, which was formed from the original groove at the stagnation point $\beta_1 - \alpha \log 2$, pinches off from the advancing interface and is left behind the moving front. The moving front for $t > t^*$ is described by the same solution but without the pair of logarithms. This

process can be seen in Figure 1. The shape of the droplet does not change after $t = t^*$. This is because of two limitations of the present theory: both the viscosity of the less viscous liquid and the surface tension were chosen to be zero. This leads the pressure along the droplet boundary to be a constant, and then, by virtue of the maximum theorem for harmonic functions, the pressure should be constant throughout the interior of the droplet. The pressure gradient acting on the droplet is zero. Consequently the droplet is at rest in the non-viscous domain. In the real situation, where neither the second liquid viscosity nor the surface tension is zero, the viscous droplet will be dragged by the flow and its shape will continue to be modified.

The shape of the droplet described by (5) has a simple form: this is an oval with a horizontal length of the order of $\beta_2 - \beta_1 + 2\alpha \log 2$ and the vertical size about $\pi\alpha$. The droplet's shape can be, however, much more complex. It can be described by the N-logarithmic solution (2) such that the α_k corresponding to the minimal $|\alpha_k|$ is chosen to be $-\sum_l' \alpha_l$, where the prime in the sum indicates summation over all l from 1 to N , but omitting $l = k$.

In [4] the geometrical interpretation of the constants of motion $\{\alpha_k\}$ and $\{\beta_k\}$ have been found. Namely, $\beta_k - \alpha_k \log 2$ is the complex coordinate of the stagnation point near which the k^{th} groove with parallel walls originates. And $\pi|\alpha_k|$ and $\arg(\alpha_k)$ are the width and the angle with respect to the horizontal axes respectively. This geometrical interpretation appeared to be in the excellent agreement with known real and numerical experiments. But this interpretation has a constraint: $|\arg(\alpha_k)| < \pi/2$. In this work we consider α_k which violates the last inequality. For such α_k the $\beta_k - \alpha_k \log 2$ is the complex coordinate of the point where the previously formed groove starts to shrink. Clearly, for the simplest example with two logarithms, where $\alpha_1 > 0$ and $\alpha_2 < 0$, the width of the groove which was originally $\pi\alpha_1$ becomes after shrinking $\pi(\alpha_1 + \alpha_2)$. Of course, this consideration is valid only when $\alpha_1 + \alpha_2 \geq 0$. In the case of equality the width vanishes completely, it corresponds to the formation of the droplet described above. We do not understand on this stage why and when the terms with $\text{Im}(\alpha_k) < 0$ are physically realizable. We hope that future studies will elucidate this point.

Let us derive the droplet's neck dynamics $w(t)$ prior to pinch off in the example (5) discussed above. The width $w(t)$ of the neck is defined as

$$w(t) = 2y(t, \phi^*(t)) = 2\text{Im } z(t, \phi^*(t)), \quad (14)$$

where ϕ^* is a non-zero solution of the equation

$$\partial y / \partial \phi = 0, \quad (15)$$

as it is clear from (5) and from Figure 1. Differentiating (5) and solving (15), having in mind that both a and b are close to 1, we obtain $\phi^* = \sqrt{\alpha(a-b)}$ in the leading

order with respect to $(a-b)$ which is small. Then from (14) and (5) it follows that

$$w(t) = 2[\phi^* + \alpha(\arctan \frac{\sin \phi^*}{\cos \phi^* - a} - \arctan \frac{\sin \phi^*}{\cos \phi^* - b})]. \quad (16)$$

Replacing here ϕ^* by its value found above we finally obtain

$$w(t) = 4\sqrt{\alpha(a-b)} = 4\sqrt{\alpha(1-c^2)q_0(t^*-t)}, \quad (17)$$

where constants c , q_0 , and t^* are given above.

Here we considered only the simplest configuration (5) leading to the formation of a single droplet. Clearly, there is a broad subclass of solutions (2) and (4) which describes in a similar fashion formation and pinching off of an arbitrary number of viscous droplets with different shapes. The same analytic continuation, performed above for the simple case $N = 3$, holds for this generalized case. Again, this continuation contains the elimination of the removable singularity by equating $(e^{i\phi} - 1)/(e^{i\phi} - 1)$ to 1 at the moment of formation of each droplet, thus decreasing the total number of logarithms in the description of the moving interface, as done above for $N = 3$. All these considerations hold for both the channel geometry and for closed interfaces.

Previous studies on topological changes in the viscous flow in the Hele-Shaw channel [6] considered surface tension as a significant factor. Because surface tension is zero in the present paper and because we believe that there are many different regimes of pinching off within classes of solutions (2) and (4), we do not compare our findings with results in [6]. There is also a paper on the change of topology in the absence of surface tension [7], which states that a change of topology is inevitable in the interior (suction) problem if the sink is not located at the center of mass of the viscous domain. Here we may say that we know how to describe possible topological changes in detail. What we do not know is which of the solutions obtained above are physically realizable and which are not. We also do not know at this stage what will be the corrections to the dynamics of droplet formations caused by non-zero surface tension. We hope that future studies will answer these questions.

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